II.
The Ibozoo uu
and
Accelerated Observers in Flat Spacetime

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Abstract

In this second paper about the Ibozoo uu or nodes of oriented arrows model, we will see how this one brings us to an accelerated movement quite similar to the hyperbolic one as long as the distance between inertial and accelerated systems is small before c^2/g where c is the speed of light and g a constant accelaration. We will see also that locally, we retrieve the "lorentzian spacetime" as in our first paper. Invariant quantities are also found but, conversely to special relativity, the minus sign is replaced by a positive one because of the cyclic character of the model.

Contents

1 Introduction

The concept of node of oriented arrows (NOA) has been introduced in a first paper[2]. A NOA is a junction of oriented (perpendicular or linearly independent) arrows or axis. They form a representation basis of a linear space. A NOA is a mathematical object on which rotation transformations (i.e. SO(n) group) are applied. For this group, the number of parameters or real angles is (n-1)n/2 which is also the number of arrows on each NOA. Rotations over a given NOA occur about arrows of another NOA which is taken as reference. For n = 3 a 1-dimensional plus 1 (1D+1) spacetime is generated by many NOA or, equivalently, by successive rotations of a NOA[2]. That spacetime shows local lorentzian behaviors, i.e. the usual time dilation and length contraction[3]. In addition, conversely to special relativity where the speed limit c is imposed, that limit is a consequence of arrows inversion (i.e. permutation[4] of time and space arrows between two NOA). Figure 1 gives a geometric representation of a 3-dimensional (i.e. 3 arrows) NOA.

In NOA model, only the parameters of SO(n) (i.e. the rotation angles of a NOA about arrows of another NOA taken as reference) are relevant. Actually, one of the main interests is to get mathematical relations among those parameters when two or more different references are taken into account. In NOA model there is no distance, time or motion among NOA but simply angles. For instance, a distance x between two geometric points in what we usually call euclidian space appears, in NOA paradigm, as a rotation of one NOA about the arrow x of another NOA. The more the angle of rotation is large, the more important is the value of x. A straight line in euclidian space is, in NOA paradigm, a chain of N NOA (i.e. N is a positive integer which goes to infinity) where each NOA is adequately rotated about the arrow x of a NOA of reference. In these two examples, there are no rotations of any NOA about arrows t and Γ_{x} of the NOA of reference. Consequently, only the x arrows of all NOA are parallel in that particular case.

The space and time arrows x and t of any NOA are respectively associated with the usual coordinates of space x and time t. The connections among arrows and coordinates is given by universal functions $\mathcal{F}$ such as:

$$x = \mathcal{F}_{\mathrm{x}}(\phi_{\mathrm{x}}) \tag{1}$$

and

$$t = \mathcal{F}_{\mathrm{t}}(\phi_{\mathrm{t}}) \tag{2}$$

which transform angles into physical coordinates where ϕ_{x} and ϕ_{t} are the angles of rotation about arrows x and t respectively with $\mathcal{F}_{\mathrm{x}}(\alpha)/\mathcal{F}_{\mathrm{t}}(\alpha) = \mathrm{c}$ where α is an arbitrary angle. Other characteristics of those functions are given in the first paper[2]. The arrow Γ_{x} on Fig. 1 is associated with the relative state of motion as shown in that paper. In [2], we limited ourselves to 3 parameters or angles and then to 3 arrows so, to the SO(3) group.

From a mathematical viewpoint, instead of the rotation group SO(3), i.e. special orthogonal 3×3 matrices for which det(...) = +1 and with 3/2(3-1) = 3 real parameters; real angles, we used in [2] its universal covering group SU(2), i.e. special unitary 2×2 matrices where det(...) = +1 with $2^2-1 = 3$ real angles, which is homomorphic onto SO(3). Indeed, SU(2) is double-valued: to each group element of SO(3) corresponds two elements of SU(2), i.e. a rotation of 0, $\pm 2\pi$, $\pm 6\pi$, ... about any axis gives the same as $\pm 4\pi$, $\pm 8\pi$, ... excepted for an opposite sign. This is the origin of the oriented entanglement relation among NOA discussed in [2].

Therefore, with SU(2), a 3D-NOA can be represented as a quaternion ξ which is a 2×2 matrix. The operations on a 3D-NOA are just the element of SU(2), i.e. real rotation operations R. If ξ_{r} represents a NOA taken as reference, say NR, then $\xi_{\mathrm{r}} = \sigma_{\mathrm{o}}$ where σ_{o} is the 2×2 unit matrix. So, any NOA ξ' can be obtained by one or many successive rotations about any arrow u which is expressed as a linear combination of the arrows x, t and Γ_{x} of NR such as $\xi' = \mathrm{R}_{\mathrm{u}}\xi_{\mathrm{r}}$. It has to be noted that this is different from the rotation of a 3-vector described in the spin-matrix language. In this language a 3-vector $\mathbf{r} = (x,\, y,\, z)$ takes the form of a quaternion X where $X = x\sigma_x + y\sigma_y + z\sigma_z$ in which the σ's are the usual Pauli's matrices and the rotation to get $\mathbf{r}'$ from $\mathbf{r}$ is given by: $X \to X' = \mathrm{R}_{\mathrm{u}}X{\mathrm{R}_{\mathrm{u}}}^{-1}$ where actually, ${\mathrm{R}_{\mathrm{u}}}^{-1} = \mathrm{R}_{\mathrm{u}}^{\star}$. From this we can say that the ξ's (i.e. 3D-NOA) are treated more like 2×2 spinors.

In the first paper we limited ourselves to non-accelerated reference frames. In this second paper it will be shown how an accelerated frame is represented in 3D-NOA model. (Note: this accelerated frame is massless. Massive case needs more dimensions or arrows see §5). After that, its motion relatively to an inertial frame will be deduced for the special case of a uniform acceleration. As long as the distance d between both frames is small before c^2/g, where c and g are respectively the speed of light and the acceleration, the time-dependent position coordinate of the accelerated frame given by this model is approximately equal to the hyperbolic one with a discrepancy less than 2% for $d \leq \mathrm{c}^2/3\mathrm{g}$ and less than 10% for $d \leq \mathrm{c}^2/\mathrm{g}$. The value c^2/g coincides, in NOA model, with the maximal value $\pi/2$ imposed to the angle $\phi_{\Gamma_{\mathrm{x}}}$ about arrow Γ_{x}. Furthermore, locally the "lorentzian spacetime" is retrieved and the NOA model shows similar invariants as special relativity (SR) except the minus sign in SR is replaced by a plus in NOA representation.

It has to be noted that conversely to the Lorentz group SO(3,1), SO(3) and SU(2) are compact groups. As the Lorentz group, SO(3) and SU(2) are continuous but, for those ones, every infinite sequences of elements has a limit element which is not the case for the Lorentz group: there is no element (i.e. transformation operation) that corresponds to the limiting value c (i.e. the boost parameter, which is a real angle, must go to infinity). In other words, with real angles, SO(3) and SU(2) behave cyclicly because of the sine and cosine functions embedded in the matrix representation of the element of those compact groups.

In Lorentz group (for real angles), there are no cyclic behaviors because the previous functions are replaced by hyperbolic sine and cosine. Consequently, the use of a compact group such SU(2) to retreive a spacetime usually generated by a noncompact one (i.e. Lorentz group) leads us to some important differences as we will see later.

2 Acceleration and Motion in NOA Model

In this section the sequence of operations of rotation needed to represent the acceleration is given. In NOA model, operations of rotation allow to pass from one NOA to another or equivalently to transform one NOA into another. For instance, a finite displacement in ordinary space is represented as a sequence of many small rotations applied to a NOA (i.e. taken as reference) about its arrow x and give a chain of NOA. In other words, each rotation transforms the preceding NOA into the next one which is rotated from the previous by a small angle about the x arrow of the reference. Similarly for time displacement, rotations are taken about the t arrow. For acceleration, the sequence is more complicated because arrows x, t and Γ_{x} must intervene together.

2.1 Definitions and Procedure

Let's consider two reference frames S and S'(see Fig. 2). S is an inertial reference frame and S' is the accelerated one. Those are in flat spacetime so, far from gravitational sources. We assume, without losses of generality, that the origins of S and S', $x = x' = 0$, coincide at the initial moment, $t = t' = 0$ where t and t' are the times coordinates indicated by their respective clocks which are located at $x = 0$ and $x' = 0$ respectively. It is also assumed that when $t = t' = 0$, their relative velocity v is zero. However, S' experiences a constant and positive acceleration along the x axis of S which is equals to g and is different from zero since $t = t' = 0$. Conversely to the first paper where S and S' were two inertial frames, here S' is a privileged one because the observer in S' can feel and measure the physical effects of its acceleration which is not the case of S. So, the observers in S and S' know that S' is the one which is really accelerated.

What we want is to find the space coordinate of the origin of S' and its time coordinate both relatively to the inertial frame S. This will give us the so-called "world line" of S'.

To do the job, in NOA representation, we have to take into account many chains of NOA. The chain concept has been introduced and defined in our first paper. Let's define those chains in this particular situation. On each chain one NOA of reference is taken. Let's call it $\mathrm{NR}^{(\mathrm{j})}$ and its chain, $\mathrm{chNR}^{(\mathrm{j})}$. The index j is the chain number; j = 0, ± 1, ± 2, ..., $\pm M$ with $M \to \infty$. $\mathrm{x}^{(\mathrm{j})}$, $\mathrm{t}^{(\mathrm{j})}$ and $\Gamma_{\mathrm{x}}^{(\mathrm{j})}$ are the arrows of $\mathrm{NR}^{(\mathrm{j})}$ on chain $\mathrm{chNR}^{(\mathrm{j})}$.

Along a given $\mathrm{chNR}^{(j)}$, each NOA (or "chain link") is characterized by three angles: $(\phi^{(p)}_{x^{(j)}}, \phi^{(p)}_{t^{(j)}}, \phi^{(p)}_{\Gamma_x{}^{(j)}})$ where the subscript is the arrow of $\mathrm{NR}^{(j)}$ about of which the rotation occurs and where p = 0, ±1, ±2, ... $\pm N$ with $N \to \infty$. p stands for the "chain link" number. For those chains we have $\phi^{(p)}_{x^{(j)}} = \phi^{(p)}_{\Gamma_x{}^{(j)}} = 0$ for all p and j. However:

$$\phi^{(p)}_{t^{(j)}} = p\Delta\alpha \tag{3}$$

for all j where $\Delta\alpha$ is a constant and very small angle. This means that all chains $\mathrm{chNR}^{(j)}$ are exclusively chains of time. In other words, the set of NOA along a given chain represents the flow of time at a fixed point in ordinary space (i.e. $\phi^{(p)}_{x^{(j)}} = 0$ relatively to $\mathrm{NR}^{(j)}$) which point, of course, has no motion (i.e. $\phi^{(p)}_{\Gamma_x{}^{(j)}} = 0$ relatively to $\mathrm{NR}^{(j)}$). $\phi^{(0)}_{t^{(j)}} = 0$ corresponds to the NOA or "link" $\mathrm{NR}^{(j)}$, i.e. the reference itself on $\mathrm{chNR}^{(j)}$. The state of each NOA on a chain $\mathrm{chNR}^{(j)}$ is defined by the three angles (0, p$\Delta\alpha$, 0) relatively to the reference $\mathrm{NR}^{(j)}$.

Let's set j = 0 for the NOA of reference associated with the origin of S (i.e. $\mathrm{NR}^{(0)}$). The chain j = 0 represents the flow of time of the origin (i.e. the clock) of S which is a fixed and an unmoving space point (or clock) relatively to S. Because we are interested to get the world line of the origin of S' relatively to the origin of S, the chain j = 0 and then the NOA $\mathrm{NR}^{(0)}$ will be our main reference.

The distinction among chains j is simply characterized by a rotation of $\mathrm{NR}^{(j)}$ about the arrow $\Gamma^{(0)}_x$ of $\mathrm{NR}^{(0)}$. More precisely, the angles of rotation of $\mathrm{NR}^{(j)}$ about the arrows $x^{(0)}$, $t^{(0)}$ and $\Gamma^{(0)}_x$ of $\mathrm{NR}^{(0)}$ are respectively (0, 0, $\gamma^{(j)}_{\Gamma^{(0)}_x}$) so, $\Gamma^{(j)}_x = \Gamma^{(0)}_x$ for all j (i.e. arrows Γ_x of all $\mathrm{NR}^{(j)}$ are parallel). We must remember that $\Gamma^{(0)}_x$ is the arrow which characterizes the state of movement relatively to S, so the angle $\gamma^{(j)}_{\Gamma^{(0)}_x}$ represents the intensity (or in some way the velocity[2]) of that movement. Note that $\gamma^{(0)}_{\Gamma^{(0)}_x} = 0$; the movement of S relatively to S is always zero. $\gamma^{(j)}_{\Gamma^{(0)}_x}$ can be written as:

$$\gamma^{(j)}_{\Gamma^{(0)}_x} = j\Delta\Omega \tag{4}$$

where $\Delta\Omega$ is a constant and very small angle as $\Delta\alpha$.

Initially, when the origins of S and S' coincide, the representative state of the origin of S and the one of the origin of S' occupy the same NOA: $\mathrm{NR}^{(0)}$. The representative state of the origin of S (i.e. the state of S to simplify) starts from $\mathrm{NR}^{(0)}$ and "jumps" from one NOA to another (increasing p) along the chain j = 0. Only its clock runs; the position and the movement of that clock doesn't change relatively to S. On the other hand, the representative state of the origin of S' (i.e. the state of S' to simplify) starts from $\mathrm{NR}^{(0)}$ and "jumps" not only along a chain (i.e. its clock is running) but also from chain to chain because the origin or the clock of S' is accelerated (i.e. its state of movement, j, changes with time, p). Consequently, the integer j becomes a function of the integer p:

$$j \to j(p) \ , \tag{5}$$

with j(0) = 0 (i.e. initial condition).

It has to be noted that each $NR^{(j)}$ represents the initial origin of an inertial frame which has a uniform movement (i.e. constant velocity) relatively to the inertial frame S (or $NR^{(0)}$). So, when the state of S' jumps on one of these chains, it is instantaneously in a inertial frame as required by special relativity. In addition, relatively to $NR^{(j)}$ on chain j, the state of S' is instantaneously at rest. Indeed, that state occupies a given NOA on that chain and all NOA on that chain experience no rotation about $\Gamma_x^{(j)}$: $\phi^{(p)}_{\Gamma_x^{(j)}} = 0$ for all p and j as mentioned above. Furthermore, each NOA on that chain has no rotation (i.e. $\phi^{(p)}_{x^{(j)}} = 0$) about the arrow $x^{(j)}$ of $NR^{(j)}$. Consequently, the position of the origin of S' is always zero relatively to S' as it has to be. Finally, because both clocks are identical, synchronized and unaffected by acceleration, the state of S and the one of S' must always occupy the same index p on their respective chain. This result has been shown in §5.2.3, equation (82), of [2].

2.2 Operations and Results

In general, when the state of S' starts from $NR^{(0)}$ and jumps from chain to chain (j) and from link to link (p), the angle $\gamma^{(j)}_{\Gamma_x^{(0)}}$ is a function of $\phi^{(p)}_{t^{(j)}}$ ($= p\Delta\alpha$) or simply a function of p, i.e. $\gamma^{(j(p))}_{\Gamma_x^{(0)}}$ because of (4)-(5). This function characterizes the time-dependent "velocity" of the origin of S' relatively to the one of S. Consequently, the variation of $\gamma^{(j)}_{\Gamma_x^{(0)}}$:

$$\Delta\gamma^{(j)}_{\Gamma_x^{(0)}} \to \Delta\gamma^{(j(p))}_{\Gamma_x^{(0)}} \ = \ \gamma^{(j(p+1))}_{\Gamma_x^{(0)}} - \gamma^{(j(p))}_{\Gamma_x^{(0)}} \ = \ [j(p+1) - j(p)]\Delta\Omega \ \equiv \ \Delta\gamma(p), \tag{6}$$

is also a function of p.

Formally, in the NOA representation, the "trajectory" of the state of the origin of S' upon the chains relatively to the $NR^{(j)}$'s (i.e. S' relatively to S' or, more exactly, relatively to an inertial frame comoving instantaneously with S') is given by the following sequence of operations:

$$\prod_{p=0}^{M_o-1} \left\{ R_{t^{(j(p+1))}}(\Delta\alpha)\ R_{\Gamma_x^{(j(p))}}(\Delta\gamma^{(j(p))}_{\Gamma_x^{(j(p))}}) \right\} = \prod_{p=0}^{M_o-1} \left\{ R_{t^{(j(p+1))}}(\Delta\alpha)\ R_{\Gamma_x^{(0)}}(\Delta\gamma^{(j(p))}_{\Gamma_x^{(0)}}) \right\} \tag{7}$$

where the integer M_o has to be found and where we used the fact that the variation of $\phi^{(p)}_{t^{(j)}} \to \phi^{(p)}_{t^{(j(p))}}$ is simply $\Delta\phi^{(p)}_{t^{(j(p))}} = \phi^{(p+1)}_{t^{(j(p))}} - \phi^{(p)}_{t^{(j(p))}} = \Delta\alpha$, according to (3). It has to be noted that we used $\Gamma_x^{(0)}$ instead of $\Gamma_x^{(j)}$ in (7) because $\Gamma_x^{(j)}$ $= \Gamma_x^{(0)}$ as mentioned above. As $\Delta\alpha$, the angle $\Delta\gamma^{(j)}_{\Gamma_x^{(0)}}$ in (7) is also infinitesimal so, the rotation operators "commute" [5]. The arrow $\mathrm{t}^{(\mathrm{j(p+1)})}$ can be expressed as linear combination of $\mathrm{t}^{(0)}$ and $\mathrm{x}^{(0)}$ which are the arrows of $\mathrm{NR}^{(0)}$ on chain $\mathrm{j} = 0$ (the origin of S):

$$\mathrm{t}^{(\mathrm{j(p+1)})} \;=\; \mathrm{t}^{(0)} \cos\left(\gamma^{(\mathrm{j(p+1)})}_{\Gamma^{(0)}_{\mathrm{x}}}\right) \;+\; \mathrm{x}^{(0)} \sin\left(\gamma^{(\mathrm{j(p+1)})}_{\Gamma^{(0)}_{\mathrm{x}}}\right) . \tag{8}$$

On the other hand, relatively to the origin of S (i.e. arrows of $\mathrm{NR}^{(0)}$), the "trajectory" of the state of the origin of S' is:

$$\prod_{\mathrm{p}=0}^{M_{\mathrm{o}}-1} \left\{ \mathrm{R}_{\mathrm{t}^{(0)}}(\Delta\theta^{(\mathrm{j(p)})}_{\mathrm{t}^{(0)}})\; \mathrm{R}_{\mathrm{x}^{(0)}}(\Delta\theta^{(\mathrm{j(p)})}_{\mathrm{x}^{(0)}})\; \mathrm{R}_{\Gamma^{(0)}_{\mathrm{x}}}(\Delta\gamma^{(\mathrm{j(p)})}_{\Gamma^{(0)}_{\mathrm{x}}}) \right\} . \tag{9}$$

All operators R in (9) "commute" [5] because all angles are infinitesimal. The variations of θ's are simply: $\Delta\theta^{(\mathrm{j(p)})}_{\mathrm{x}^{(0)}} = \theta^{(\mathrm{j(p+1)})}_{\mathrm{x}^{(0)}} - \theta^{(\mathrm{j(p)})}_{\mathrm{x}^{(0)}}$ and $\Delta\theta^{(\mathrm{j(p)})}_{\mathrm{t}^{(0)}} = \theta^{(\mathrm{j(p+1)})}_{\mathrm{t}^{(0)}} - \theta^{(\mathrm{j(p)})}_{\mathrm{t}^{(0)}}$. As for $\gamma^{(\mathrm{j})}_{\Gamma^{(0)}_{\mathrm{x}}}$, all $\theta^{(\mathrm{j})}_{\mathrm{x}^{(0)}}$ and $\theta^{(\mathrm{j})}_{\mathrm{t}^{(0)}}$ are functions of $\phi^{(\mathrm{p})}_{\mathrm{t}^{(\mathrm{j})}}$ or p. In other words, they are functions of the proper time of S'. These functions are the ones that will give us the world line of S' relatively to S. Note that $\theta^{(0)}_{\mathrm{x}^{(0)}} = \theta^{(0)}_{\mathrm{t}^{(0)}} = 0$. These last ones are the angular coordinates of S' relatively to S (i.e. about arrows $\mathrm{x}^{(0)}$ and $\mathrm{t}^{(0)}$ of $\mathrm{NR}^{(0)}$) when the origins of S and S' coincide.

The two "trajectories" must be the same, i.e. they are simply two viewpoints of the same state of the origin of S' so:

$$\begin{aligned} \prod_{\mathrm{p}=0}^{M_{\mathrm{o}}-1} \Big\{ & \mathrm{R}_{\mathrm{t}^{(\mathrm{j(p+1)})}}(\Delta\alpha)\; \mathrm{R}_{\Gamma^{(0)}_{\mathrm{x}}}(\Delta\gamma^{(\mathrm{j(p)})}_{\Gamma^{(0)}_{\mathrm{x}}}) \\ & - \mathrm{R}_{\mathrm{t}^{(0)}}(\Delta\theta^{(\mathrm{j(p)})}_{\mathrm{t}^{(0)}})\; \mathrm{R}_{\mathrm{x}^{(0)}}(\Delta\theta^{(\mathrm{j(p)})}_{\mathrm{x}^{(0)}})\; \mathrm{R}_{\Gamma^{(0)}_{\mathrm{x}}}(\Delta\gamma^{(\mathrm{j(p)})}_{\Gamma^{(0)}_{\mathrm{x}}}) \Big\} \;=\; 0 . \end{aligned} \tag{10}$$

As shown in [2], infinitesimal angles are needed to satisfy exactly an equality like (10). According to eqs. (92)-(95) in [2], we have for small angles:

$$\mathrm{R}_{\mathrm{x}^{(0)}}(\Delta\theta^{(\mathrm{j})}_{\mathrm{x}^{(0)}}) \;\simeq\; \sigma_{\mathrm{o}} - \mathrm{i}\frac{\Delta\theta^{(\mathrm{j})}_{\mathrm{x}^{(0)}}}{2}\,\sigma_{\mathrm{x}} , \tag{11}$$

$$\mathrm{R}_{\mathrm{t}^{(0)}}(\Delta\theta^{(\mathrm{j})}_{\mathrm{t}^{(0)}}) \;\simeq\; \sigma_{\mathrm{o}} - \mathrm{i}\frac{\Delta\theta^{(\mathrm{j})}_{\mathrm{t}^{(0)}}}{2}\,\sigma_{\mathrm{t}} , \tag{12}$$

$$\mathrm{R}_{\Gamma^{(0)}_{\mathrm{x}}}(\Delta\gamma^{(\mathrm{j})}_{\Gamma^{(0)}_{\mathrm{x}}}) \;\simeq\; \sigma_{\mathrm{o}} - \mathrm{i}\frac{\Delta\gamma^{(\mathrm{j})}_{\Gamma^{(0)}_{\mathrm{x}}}}{2}\,\sigma_{\Gamma_{\mathrm{x}}} \tag{13}$$

and

$$\mathrm{R}_{\mathrm{t}^{(\mathrm{j})}}(\Delta\alpha) \;\simeq\; \sigma_{\mathrm{o}} - \mathrm{i}\frac{\Delta\alpha}{2}\Big(\sin\left(\gamma^{(\mathrm{j})}_{\Gamma^{(0)}_{\mathrm{x}}}\right)\,\sigma_{\mathrm{x}} + \cos\left(\gamma^{(\mathrm{j})}_{\Gamma^{(0)}_{\mathrm{x}}}\right)\,\sigma_{\mathrm{t}} \Big) , \tag{14}$$

where[2]:

$$\sigma_{\Gamma_{\mathrm{x}}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{\mathrm{t}} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_{\mathrm{x}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_{\mathrm{o}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{15}$$

Therefore:

$$\prod_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Big\{\mathrm{R}_{\mathrm{t}^{(0)}}(\Delta\theta_{\mathrm{t}^{(0)}}^{(\mathrm{j(p)})})\ \mathrm{R}_{\mathrm{x}^{(0)}}(\Delta\theta_{\mathrm{x}^{(0)}}^{(\mathrm{j(p)})})\ \mathrm{R}_{\Gamma_{\mathrm{x}}^{(0)}}(\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})})\Big\}\ \simeq$$
$$\prod_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Big\{\sigma_{\mathrm{o}}-\tfrac{\mathrm{i}}{2}\Delta\theta_{\mathrm{x}^{(0)}}^{(\mathrm{j(p)})}\sigma_{\mathrm{x}}-\tfrac{\mathrm{i}}{2}\Delta\theta_{\mathrm{t}^{(0)}}^{(\mathrm{j(p)})}\sigma_{\mathrm{t}}-\tfrac{\mathrm{i}}{2}\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})}\sigma_{\Gamma_{\mathrm{x}}}$$
$$+\ \Theta\Big[\Delta\theta_{\mathrm{x}^{(0)}}^{(\mathrm{j(p)})}\cdot\Delta\theta_{\mathrm{t}^{(0)}}^{(\mathrm{j(p)})}\Big]+\ \Theta\Big[\Delta\theta_{\mathrm{x}^{(0)}}^{(\mathrm{j(p)})}\cdot\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})}\Big]+\ \Theta\Big[\Delta\theta_{\mathrm{t}^{(0)}}^{(\mathrm{j(p)})}\cdot\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})}\Big]\Big\}\ . \tag{16}$$

The last three terms, the Θ's, are nonlinear functions of the small (i.e. infinitesimal) angles, i.e. products of them. Keeping only the linear (dominant) contributions we get:

$$\prod_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Big\{\mathrm{R}_{\mathrm{t}^{(0)}}(\Delta\theta_{\mathrm{t}^{(0)}}^{(\mathrm{j(p)})})\ \mathrm{R}_{\mathrm{x}^{(0)}}(\Delta\theta_{\mathrm{x}^{(0)}}^{(\mathrm{j(p)})})\ \mathrm{R}_{\Gamma_{\mathrm{x}}^{(0)}}(\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})})\Big\}\ \simeq$$
$$\sigma_{\mathrm{o}}-\tfrac{\mathrm{i}}{2}\Big(\sum_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Delta\theta_{\mathrm{x}^{(0)}}^{(\mathrm{j(p)})}\Big)\sigma_{\mathrm{x}}-\tfrac{\mathrm{i}}{2}\Big(\sum_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Delta\theta_{\mathrm{t}^{(0)}}^{(\mathrm{j(p)})}\Big)\sigma_{\mathrm{t}}-\tfrac{\mathrm{i}}{2}\Big(\sum_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})}\Big)\sigma_{\Gamma_{\mathrm{x}}}\ . \tag{17}$$

On the other hand:

$$\prod_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Big\{\mathrm{R}_{\mathrm{t}^{(\mathrm{j(p+1)})}}(\Delta\alpha)\ \mathrm{R}_{\Gamma_{\mathrm{x}}^{(0)}}(\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})})\Big\}\ \simeq$$
$$\prod_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Big\{\sigma_{\mathrm{o}}-\tfrac{\mathrm{i}}{2}\Delta\alpha\sin(\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p+1)})})\ \sigma_{\mathrm{x}}-\tfrac{\mathrm{i}}{2}\Delta\alpha\cos(\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p+1)})})\ \sigma_{\mathrm{t}}-\tfrac{\mathrm{i}}{2}\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})}\sigma_{\Gamma_{\mathrm{x}}}$$
$$+\ \Theta\Big[\Delta\alpha^{2}\Big]+\ \Theta\Big[\Delta\alpha\cdot\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})}\Big]\Big\}\ . \tag{18}$$

As before, we take only the linear contributions, so:

$$\prod_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Big\{\mathrm{R}_{\mathrm{t}^{(\mathrm{j(p+1)})}}(\Delta\alpha)\ \mathrm{R}_{\Gamma_{\mathrm{x}}^{(0)}}(\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})})\Big\}\ \simeq$$
$$\sigma_{\mathrm{o}}-\tfrac{\mathrm{i}}{2}\Big(\sum_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Delta\alpha\sin(\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p+1)})})\Big)\sigma_{\mathrm{x}}-\tfrac{\mathrm{i}}{2}\Big(\sum_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Delta\alpha\cos(\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p+1)})})\Big)\sigma_{\mathrm{t}}$$
$$-\tfrac{\mathrm{i}}{2}\Big(\sum_{\mathrm{p=0}}^{M_{\mathrm{o}}-1}\Delta\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p)})}\Big)\sigma_{\Gamma_{\mathrm{x}}}\ . \tag{19}$$

First of all, if we compare (17) with (19) we must have:

$$\Delta\theta_{\mathrm{x}^{(0)}}^{(\mathrm{j(p)})}\ =\ \Delta\alpha\ \sin\Big(\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p+1)})}\Big) \tag{20}$$

$$\Delta\theta_{\mathrm{t}^{(0)}}^{(\mathrm{j(p)})}\ =\ \Delta\alpha\ \cos\Big(\gamma_{\Gamma_{\mathrm{x}}^{(0)}}^{(\mathrm{j(p+1)})}\Big)\ . \tag{21}$$

These equations are the same as those found in [2] (i.e. eq. (83)) for infinitesimal rotations. Only the angular symbols are different. As shown in [2], the form of these equations is the "signature" of a spacetime locally "lorentzian".

Secondly, according to the definitions of $\Delta\theta^{(\mathrm{j})}_{\mathrm{x}^{(0)}}$ and $\Delta\theta^{(\mathrm{j})}_{\mathrm{t}^{(0)}}$ one has, with the initial condition $\mathrm{j}(0) = 0$:

$$\sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \Delta\theta^{(\mathrm{j}(\mathrm{p}))}_{\mathrm{x}^{(0)}} \;=\; \theta^{(\mathrm{j}(M_\mathrm{o}))}_{\mathrm{x}^{(0)}} - \theta^{(0)}_{\mathrm{x}^{(0)}} \;, \tag{22}$$

$$\sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \Delta\theta^{(\mathrm{j}(\mathrm{p}))}_{\mathrm{t}^{(0)}} \;=\; \theta^{(\mathrm{j}(M_\mathrm{o}))}_{\mathrm{t}^{(0)}} - \theta^{(0)}_{\mathrm{t}^{(0)}} \;. \tag{23}$$

and consequently:

$$\theta^{(\mathrm{j}(M_\mathrm{o}))}_{\mathrm{x}^{(0)}} - \theta^{(0)}_{\mathrm{x}^{(0)}} \;=\; \sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \Delta\alpha \sin(\gamma^{(\mathrm{j}(\mathrm{p}+1))}_{\Gamma^{(0)}_\mathrm{x}}) \;=\; \sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \Delta\alpha \sin\left(\gamma^{(0)}_{\Gamma^{(0)}_\mathrm{x}} + \sum_{\mathrm{n}=0}^{\mathrm{p}} \Delta\gamma^{(\mathrm{j}(\mathrm{n}))}_{\Gamma^{(0)}_\mathrm{x}}\right) \tag{24}$$

$$\theta^{(\mathrm{j}(M_\mathrm{o}))}_{\mathrm{t}^{(0)}} - \theta^{(0)}_{\mathrm{t}^{(0)}} \;=\; \sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \Delta\alpha \cos(\gamma^{(\mathrm{j}(\mathrm{p}+1))}_{\Gamma^{(0)}_\mathrm{x}}) \;=\; \sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \Delta\alpha \cos\left(\gamma^{(0)}_{\Gamma^{(0)}_\mathrm{x}} + \sum_{\mathrm{n}=0}^{\mathrm{p}} \Delta\gamma^{(\mathrm{j}(\mathrm{n}))}_{\Gamma^{(0)}_\mathrm{x}}\right) \tag{25}$$

for finite rotations. The definition of $\Delta\gamma^{(\mathrm{j})}_{\Gamma^{(0)}_\mathrm{x}}$ in (6) has been used to write the last equalities. Now, according to (6), let's define the "angular" acceleration as:

$$g_\gamma(\mathrm{p}) \;\equiv\; \frac{\Delta\gamma^{(\mathrm{j}(\mathrm{p}))}_{\Gamma^{(0)}_\mathrm{x}}}{\Delta\alpha} \;=\; \frac{\Delta\gamma(\mathrm{p})}{\Delta\alpha} \;. \tag{26}$$

Because of that, eqs.(24)-(25) can be rewritten as:

$$\theta^{(\mathrm{j}(M_\mathrm{o}))}_{\mathrm{x}^{(0)}} - \theta^{(0)}_{\mathrm{x}^{(0)}} \;=\; \Delta\alpha \sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \sin\left(\gamma^{(0)}_{\Gamma^{(0)}_\mathrm{x}} + \Delta\alpha \sum_{\mathrm{n}=0}^{\mathrm{p}} g_\gamma(\mathrm{n})\right) \tag{27}$$

$$\theta^{(\mathrm{j}(M_\mathrm{o}))}_{\mathrm{t}^{(0)}} - \theta^{(0)}_{\mathrm{t}^{(0)}} \;=\; \Delta\alpha \sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \cos\left(\gamma^{(0)}_{\Gamma^{(0)}_\mathrm{x}} + \Delta\alpha \sum_{\mathrm{n}=0}^{\mathrm{p}} g_\gamma(\mathrm{n})\right) \;. \tag{28}$$

2.3 World Line of a Uniformly Accelerated Frame

As mentioned previously, we limit ourselves to time-independent or constant accelerations. In that case, the variation of j over p is simply given by:

$$\mathrm{j}(\mathrm{p}) = \zeta \mathrm{p} \tag{29}$$

where ζ is a constant integer. By using (6) and (29), the "angular" acceleration $g_\gamma(\mathrm{p})$ becomes a constant:

$$g_\gamma(\mathrm{p}) \;=\; g_\gamma \;=\; \zeta \frac{\Delta\Omega}{\Delta\alpha} \;. \tag{30}$$

In addition, let's set:

$$\mathrm{j}(M_\mathrm{o}) \;=\; \zeta M_\mathrm{o} \;\equiv\; N_\mathrm{o}\ . \tag{31}$$

In that case and using the initial conditions given earlier, i.e. $\theta^{(0)}_{\mathrm{x}^{(0)}} = \theta^{(0)}_{\mathrm{t}^{(0)}} = \gamma^{(0)}_{\Gamma^{(0)}_\mathrm{x}}$ $= 0$, the eqs.(27)-(28) take the form:

$$\theta^{(N_\mathrm{o})}_{\mathrm{x}^{(0)}} \;=\; \Delta\alpha \sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \sin\Big((\mathrm{p}+1)g_\gamma\Delta\alpha\Big) \;=\; \Delta\alpha \sum_{\mathrm{q}=1}^{M_\mathrm{o}} \sin\Big(g_\gamma \mathrm{q}\Delta\alpha\Big) \tag{32}$$

$$\theta^{(N_\mathrm{o})}_{\mathrm{t}^{(0)}} \;=\; \Delta\alpha \sum_{\mathrm{p}=0}^{M_\mathrm{o}-1} \cos\Big((\mathrm{p}+1)g_\gamma\Delta\alpha\Big) \;=\; \Delta\alpha \sum_{\mathrm{q}=1}^{M_\mathrm{o}} \cos\Big(g_\gamma \mathrm{q}\Delta\alpha\Big) \tag{33}$$

where we set q = p+1. One can verify that:

$$\sum_{\mathrm{q}=1}^{M_\mathrm{o}} \sin\Big(g_\gamma \mathrm{q}\Delta\alpha\Big) \;=\; \frac{\sin\Big(\frac{M_\mathrm{o}}{2}\ g_\gamma\Delta\alpha\Big)\sin\Big(\frac{M_\mathrm{o}+1}{2}\ g_\gamma\Delta\alpha\Big)}{\sin(g_\gamma\Delta\alpha/2)}\ , \tag{34}$$

$$\sum_{\mathrm{q}=1}^{M_\mathrm{o}} \cos\Big(g_\gamma \mathrm{q}\Delta\alpha\Big) \;=\; \frac{\sin\Big((M_\mathrm{o}+\frac{1}{2})\ g_\gamma\Delta\alpha\Big)}{2\sin(g_\gamma\Delta\alpha/2)} \;-\; \frac{1}{2}\ . \tag{35}$$

Before going further, let us introduce α which is defined by:

$$\alpha \;=\; M_\mathrm{o}\Delta\alpha\ . \tag{36}$$

According to [2], time is obtained by the application of the universal function $\mathcal{F}_\mathrm{t}$ on a time angle which is α here:

$$\tau \;\equiv\; \mathcal{F}_\mathrm{t}(\alpha) \;=\; \mathcal{F}_\mathrm{t}(M_\mathrm{o}\Delta\alpha) \;=\; M_\mathrm{o}\mathcal{F}_\mathrm{t}(\Delta\alpha) \;\equiv\; M_\mathrm{o}\Delta\tau\ . \tag{37}$$

τ is the proper time of S'. In addition, from (83) and (85) in [2] we must have:

$$\sin(g_\gamma\Delta\alpha) \;=\; \frac{\Delta\mathrm{v}}{\mathrm{c}}\ . \tag{38}$$

where Δv is an infinitesimal velocity increment of the accelerated system S'. We must remember that g_γ is a finite quantity and $\Delta\alpha$ is an infinitesimal one. Consequently:

$$\sin(g_\gamma\Delta\alpha) \;\simeq\; g_\gamma\Delta\alpha \;=\; \frac{\Delta\mathrm{v}}{\mathrm{c}} \;=\; \frac{\mathrm{g}}{\mathrm{c}}\Delta\tau \tag{39}$$

where

$$\mathrm{g} \;=\; \frac{\Delta\mathrm{v}}{\Delta\tau}\ , \tag{40}$$

is the usual acceleration. Using (37) and (39) in (34)-(35), equations (32)-(33) are rewritten as:

$$\theta^{(N_o)}_{\mathrm{x}^{(0)}} \;=\; \tfrac{\Delta\alpha}{\mathrm{g}\Delta\tau}\Big\{2\mathrm{c}\sin\left(\tfrac{\mathrm{g}}{2\mathrm{c}}\tau\right)\;\sin\left(\tfrac{\mathrm{g}}{2\mathrm{c}}(\tau+\Delta\tau)\right)\Big\} \;\simeq\; \tfrac{\Delta\alpha}{\Delta\tau}\Big\{\tfrac{2\mathrm{c}}{\mathrm{g}}\sin^2\left(\tfrac{\mathrm{g}}{2\mathrm{c}}\tau\right)\Big\} \tag{41}$$

$$\theta^{(N_o)}_{\mathrm{t}^{(0)}} \;=\; \Delta\alpha\Big\{-\tfrac{1}{2} \;+\; \tfrac{\mathrm{c}}{\mathrm{g}\Delta\tau}\sin\left(\tfrac{\mathrm{g}}{\mathrm{c}}(\tau+\tfrac{1}{2}\Delta\tau)\right) \;\simeq\; \tfrac{\Delta\alpha}{\Delta\tau}\Big\{\tfrac{\mathrm{c}}{\mathrm{g}}\sin\left(\tfrac{\mathrm{g}}{\mathrm{c}}\tau\right)\Big\}\;. \tag{42}$$

To get the last results we used the fact that, as $\Delta\alpha$, $\Delta\tau$ is an infinitesimal quantity. Finally, knowing that the position x and time t of S' relatively to S are:

$$x \;\equiv\; \mathcal{F}_{\mathrm{x}}(\theta^{(N_o)}_{\mathrm{x}^{(0)}})\;, \tag{43}$$

$$t \;\equiv\; \mathcal{F}_{\mathrm{t}}(\theta^{(N_o)}_{\mathrm{t}^{(0)}}) \tag{44}$$

and taking into account some facts about functions $\mathcal{F}$ (see first paper[2]) and also: $\mathcal{F}_{\mathrm{x}}(...) = \mathrm{c}\mathcal{F}_{\mathrm{t}}(...)$, one has with (41)-(42) and $\Delta\tau = \mathcal{F}_{\mathrm{t}}(\Delta\alpha)$:

$$x \;=\; \tfrac{2\mathrm{c}^2}{\mathrm{g}}\sin^2\left(\tfrac{\mathrm{g}}{2\mathrm{c}}\tau\right) \tag{45}$$

$$t \;=\; \tfrac{\mathrm{c}}{\mathrm{g}}\sin\left(\tfrac{\mathrm{g}}{\mathrm{c}}\tau\right)\;. \tag{46}$$

Together, these equations form the world line of S' relatively to S. One can verify that:

$$\left(x-\frac{\mathrm{c}^2}{\mathrm{g}}\right)^2 \;+\; (\mathrm{c}t)^2 \;=\; \frac{\mathrm{c}^4}{\mathrm{g}^2}\;. \tag{47}$$

This is a circular motion. See section §4 for details. Note that if $\mathrm{g}\tau/\mathrm{c}$ is a small quantity, equations (45)-(46) reduce to:

$$x \;\simeq\; \tfrac{\mathrm{g}}{2}\tau^2 \tag{48}$$

$$t \;\simeq\; \tau \tag{49}$$

and then:

$$x \;\simeq\; \tfrac{\mathrm{g}}{2}\mathrm{t}^2\;. \tag{50}$$

This is the non-relativistic result well known in classical mechanics for initial velocity and position equal to zero.

2.4 Invariants

Let us introduce the 4-velocity vector (actually 2-velocity because we are limited to one space coordinate, 1D, and time) with, as usual, the superscript "0" for time and "1" for the space coordinate:

$$\tilde{\mathbf{u}} \;=\; (c\tilde{u}^{0}, \tilde{u}^{1}) \tag{51}$$

and the 4-acceleration (actually 2-acceleration) vector:

$$\tilde{\mathbf{a}} \;=\; (c\tilde{a}^{0}, \tilde{a}^{1}) \;=\; \frac{d\tilde{\mathbf{u}}}{d\tau} \;=\; (c\frac{d\tilde{u}^{0}}{d\tau}, \frac{d\tilde{u}^{1}}{d\tau}) \tag{52}$$

In NOA model we have the three following invariants:

$$\tilde{\mathbf{u}}^{2} \;=\; (\tilde{u}^{1})^{2} \;+\; (c\tilde{u}^{0})^{2} \;=\; c^{2} \tag{53}$$

$$\tilde{\mathbf{a}}^{2} \;=\; (\tilde{a}^{1})^{2} \;+\; (c\tilde{a}^{0})^{2} \;=\; g^{2} \tag{54}$$

$$\tilde{\mathbf{a}} \cdot \tilde{\mathbf{u}} \;=\; \tilde{a}^{1}\tilde{u}^{1} \;+\; c^{2}\tilde{a}^{0}\tilde{u}^{0} \;=\; 0\,. \tag{55}$$

Note the sign "+" before quantities with the superscript "0", a consequence of a rotation group transformations.

Relatively to S' or, more exactly, relatively to an inertial frame comoving instantaneously with S' (i.e. the $NR^{(j)}$ references), the 2-velocity and the 2-acceleration components of the origin of S' are:

$$\tilde{u}^{0} \;=\; \frac{d\tau}{d\tau} \;=\; 1\,, \tag{56}$$

$$\tilde{u}^{1} \;=\; \frac{dx'}{d\tau} \;=\; 0\,, \tag{57}$$

$$\tilde{a}^{0} \;=\; \frac{d\tilde{u}^{0}}{d\tau} \;=\; 0\,, \tag{58}$$

$$\tilde{a}^{1} \;=\; \frac{d\tilde{u}^{1}}{d\tau} \;=\; g\,. \tag{59}$$

$\tilde{u}^{1}$ in (57) is 0 because, previously, we mentioned in section §2.1 that relatively to $NR^{(j)}$ on chain j, the state of the origin of S' is instantaneously at rest; that state occupies a given NOA on that chain and all NOA on that chain experience no rotation about $\Gamma_x^{(j)}$: $\phi^{(p)}_{\Gamma_x^{(j)}} = 0$ for all p and j. However, the acceleration $\tilde{a}^{1}$ is different from zero because when the state of the origin of S' jump from chain to chain it experiences a change $\Delta\gamma^{(j)}_{\Gamma_x^{(0)}}$ between chains.

On the other hand, according to (45)-(46), the 2-velocity and the 2-acceleration components of the origin of S' relatively to S are given by:

$$\tilde{\mathrm{u}}^0 \ = \ \frac{\mathrm{d}t}{\mathrm{d}\tau} \ = \ \cos(\frac{\mathrm{g}}{\mathrm{c}}\ \tau) \ , \tag{60}$$

$$\tilde{\mathrm{u}}^1 \ = \ \frac{\mathrm{d}x}{\mathrm{d}\tau} \ = \ \mathrm{c}\sin(\frac{\mathrm{g}}{\mathrm{c}}\ \tau) \ , \tag{61}$$

$$\tilde{\mathrm{a}}^0 \ = \ \frac{\mathrm{d}\tilde{\mathrm{u}}^0}{\mathrm{d}\tau} \ = \ -\frac{\mathrm{g}}{\mathrm{c}}\sin(\frac{\mathrm{g}}{\mathrm{c}}\ \tau) \ = \ -\frac{\mathrm{g}}{\mathrm{c}^2}\tilde{\mathrm{u}}^1 \ , \tag{62}$$

$$\tilde{\mathrm{a}}^1 \ = \ \frac{\mathrm{d}\tilde{\mathrm{u}}^1}{\mathrm{d}\tau} \ = \ \mathrm{g}\cos(\frac{\mathrm{g}}{\mathrm{c}}\ \tau) \ = \mathrm{g}\tilde{\mathrm{u}}^0 \ . \tag{63}$$

It is easy to verify that equations (60)-(63) satisfy to (53)-(55) as (56)-(59). Finally, from (60) and (61) the squared length of separation ds^2 is also an invariant with the "+" sign:

$$\mathrm{ds}^2 \ \equiv \ (\mathrm{c}\mathrm{d}\tau)^2 \ = \ (\mathrm{d}x)^2 \ + \ (\mathrm{cd}t)^2 \ . \tag{64}$$

This result is actually a direct consequence of (20)-(21).

3 Hyperbolic Motion

Let's find the usual solution to a uniformly accelerated motion which is the one we can get by using special relativity[6]. Let's consider two reference frames S and S' shown on Fig. 2. When the origins of S and S' coincide, their respective clocks, fixed at these origins, are set to zero. The initial (relative) speed v is also taken to zero. Only the acceleration g of S' is different from zero. On Fig. 2, coordinates x and x' represent the usual space ones. As before, the motion is limited to one dimension in space within a flat spacetime.

The observer in S can fixe a clock at each coordinate x as any inertial observer can do. Each clock is identical to each other and all are synchronized in such a way that when the origins of S and S' cross each other, all of them mark same value which is zero. Because S' is accelerated, its observer cannot do the same. However, special relativity tells us that at each moment, S' is instantaneously at rest relatively to a third reference frame S" which is inertial (i.e. local inertial system) and for which its origin coincides with the one of S'. So, theoretically many synchronized clocks can be used in S" as in S. Of course, marks on clocks of S" coincide with the one at the origin of S' which is not zero in general except when the origins of S and S' coincide. To get the solution we will use the invariants approach as in [6].

3.1 Invariants

The 4-velocity vector (actually 2-velocity) with the superscript "0" for time and "1" for the space coordinate is:

$$\mathbf{u} \;=\; (\mathrm{c}\mathrm{u}^0, \mathrm{u}^1) \tag{65}$$

and the 4-acceleration (2-acceleration) vector is:

$$\mathbf{a} \;=\; (\mathrm{c}\mathrm{a}^0, \mathrm{a}^1) \;=\; \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau} \;=\; (\mathrm{c}\frac{\mathrm{d}\mathrm{u}^0}{\mathrm{d}\tau}, \frac{\mathrm{d}\mathrm{u}^1}{\mathrm{d}\tau}) \;. \tag{66}$$

In special relativity we have the three following invariants:

$$\mathbf{u}^2 \;=\; (\mathrm{u}^1)^2 \;-\; (\mathrm{c}\mathrm{u}^0)^2 \;=\; -\mathrm{c}^2 \tag{67}$$

$$\mathbf{a}^2 \;=\; (\mathrm{a}^1)^2 \;-\; (\mathrm{c}\mathrm{a}^0)^2 \;=\; \mathrm{g}^2 \tag{68}$$

$$\mathbf{a}\cdot\mathbf{u} \;=\; \mathrm{a}^1\mathrm{u}^1 \;-\; \mathrm{c}^2\mathrm{a}^0\mathrm{u}^0 \;=\; 0 \;. \tag{69}$$

Note the sign "-" before quantities with the superscript "0", a consequence of the Lorentz group transformations. It is also well known that the squared length separation ds^2 is an invariant and is given by:

$$\mathrm{ds}^2 \;\equiv\; (\mathrm{c}\mathrm{d}\tau)^2 \;=\; (\mathrm{d}x)^2 \;-\; (\mathrm{c}\mathrm{d}t)^2 \;. \tag{70}$$

3.2 World Line of a Uniformly Accelerated Frame

Relatively to S', i.e. relatively to an inertial frame S" comoving instantaneously with S', the 2-velocity and the 2-acceleration components of the origin of S' are:

$$\mathrm{u}^0 \;=\; \frac{\mathrm{d}\tau}{\mathrm{d}\tau} \;=\; 1 \;, \tag{71}$$

$$\mathrm{u}^1 \;=\; \frac{\mathrm{d}x'}{\mathrm{d}\tau} \;=\; 0 \;, \tag{72}$$

$$\mathrm{a}^0 \;=\; \frac{\mathrm{d}\mathrm{u}^0}{\mathrm{d}\tau} \;=\; 0 \;, \tag{73}$$

$$\mathrm{a}^1 \;=\; \frac{\mathrm{d}\mathrm{u}^1}{\mathrm{d}\tau} \;=\; \mathrm{g} \;. \tag{74}$$

Of course, these equations satisfy the invariants (67)-(69). But relatively to S, one can verify that the 2-acceleration vector, **a**, given by:

$$\mathrm{a}^0 \;=\; \frac{\mathrm{d}\mathrm{u}^0}{\mathrm{d}\tau} \;=\; \frac{\mathrm{g}}{\mathrm{c}^2}\mathrm{u}^1 \;, \qquad \mathrm{a}^1 \;=\; \frac{\mathrm{d}\mathrm{u}^1}{\mathrm{d}\tau} \;=\; \mathrm{g}\mathrm{u}^0 \tag{75}$$

satisfies to (69). It also satisfies to (68) by taking (67) into account. Now, to solve for $\mathbf{u}$ we can take the time-derivative of (75) and we get:

$$\frac{\mathrm{d}^2\mathrm{u}^0}{\mathrm{d}\tau^2} = \frac{\mathrm{g}^2}{\mathrm{c}^2}\mathrm{u}^0 , \qquad \frac{\mathrm{d}^2\mathrm{u}^1}{\mathrm{d}\tau^2} = \frac{\mathrm{g}^2}{\mathrm{c}^2}\mathrm{u}^1 . \tag{76}$$

Knowing that:

$$\mathrm{u}^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau} , \qquad \mathrm{u}^1 = \frac{\mathrm{d}x}{\mathrm{d}\tau} \tag{77}$$

and using the same initial conditions as in section §2.1: $x = \mathrm{u}^1 = 0$ and $\mathrm{a}^1 = \mathrm{g}$ at $\tau = t = 0$, we finally find:

$$x = \tfrac{2\mathrm{c}^2}{\mathrm{g}} \sinh^2\left(\tfrac{\mathrm{g}}{2\mathrm{c}}\tau\right) \tag{78}$$

$$t = \tfrac{\mathrm{c}}{\mathrm{g}} \sinh\left(\tfrac{\mathrm{g}}{\mathrm{c}}\tau\right) . \tag{79}$$

Compare with (45) and (46). One can verify that:

$$\left(x + \frac{\mathrm{c}^2}{\mathrm{g}}\right)^2 - (\mathrm{c}t)^2 = \frac{\mathrm{c}^4}{\mathrm{g}^2} . \tag{80}$$

This is the world line of S' relatively to S; hyperbolic motion. Compare (80) to (47). As before, if $\mathrm{g}\tau/\mathrm{c}$ is a small quantity, equations (78)-(79) reduce to:

$$x \simeq \tfrac{\mathrm{g}}{2}\tau^2 \tag{81}$$

$$t \simeq \tau \tag{82}$$

and then:

$$x \simeq \tfrac{\mathrm{g}}{2}\mathrm{t}^2 , \tag{83}$$

the same as (50).

4 Comparison

As mentioned in introduction, the presence of hyperbolic functions (i.e. in (78)-(79)) is the signature of a non-compact group; the Lorentz group of transformations. Indeed, by taking the derivatives of (78)-(79) one can easily show that the ("hyperbolic") speed of S' relatively to S is:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{c}\tanh\left(\frac{\mathrm{g}}{\mathrm{c}}\tau\right) \tag{84}$$

and that speed will reach c only when the "angle" (i.e. boost parameter) goes to infinity: $g\tau/c \to \infty$. In that case, the group parameter doesn't exist; non-compact. But the cyclic functions in (45)-(46) is the signature of a compact group; rotation group. The ("circular") speed of S' relatively to S is:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{c}\tan(\frac{\mathrm{g}}{\mathrm{c}}\tau) \tag{85}$$

which is equal to c at $g\tau/c = \pi/4$ (i.e. a finite group parameter exist; compact group) and becomes infinite at $g\tau/c = \pi/2$.

Therefore, a priori it is clear that, in general, those two groups cannot lead us to the same results; i.e. circular motion versus hyperbolic one as we can see on Fig. 3 which shows the two world lines. Of course, the circular one fails when $g\tau/c$ becomes too large (i.e. relativistic regime). Figure 4 shows the relative discrepancy in percentage between the two results for space coordinate x (i.e. (45) and (78)) and for the time t (i.e. (46) and (79)).

However, surprisingly, the compact group used within the NOA model gives good results for the non-relativistic regime (i.e. the $gt^2/2$ region on Fig. 3). We have to say that for the most usual accelerations g that we can manage on earth, except for electron accelerators, there are practically no differences between hyperbolic and circular motions. As we can see on Fig. 3, as long as $x \ll l_o$, where $l_o = c^2/g$ is a characteristic length, we stay in non-relativistic regime. For instance with $g = 9.8\mathrm{m/s}^2$, the local earth acceleration, we have $l_o = 9.1\times10^{12}$km $\simeq$ 1 light year. However, for electrons linear accelerators[7], the acceleration is such that after only $l_o \sim 10$ feet ($\sim$ 3m), the regime is relativistic: $v/c =$ 99.9%. In that case the "circular" results fail. Note that in such accelerators the electron cloud dimension (i.e. electrons bunch cloud) is only about 1 inch (2.5cm). Consequently, its position is relatively well defined and monitored because the accelerator length scales as km (2 miles or 3.2 km).

5 Discussion

Because of the failure of the 3D-NOA model in relativistic regime, a fundamental question occurs. What is (are) the cause(s) of its failure in this regime and why it gives us good results in non-relativistic one?

To get answers to those questions, we must consider this. In the 3D-NOA model (i.e. 3 arrows), we used the compact group SU(2) with real group parameters (i.e. real angles). According to Ummite texts we must stay with compact group (i.e. rotations group) in order to get cyclic behaviors as mentioned in the first paper. Is it possible to get the results of a non-compact group (i.e. hyperbolic motion) by using compact one within the NOA model? We saw that this model lead us to series (i.e. eqs. (34)-(35)) of circular functions. But this is not enough to represent any kind of functions, specially hyperbolic ones; these series are not Fourier integrals. So, something is missing in this model. But the only things we can add are new arrows; more dimensions. More dimensions means a rotation group like SO(n) with $n > 3$ or SU(n) with $n > 2$ and real angles. This seems hopeless because we face compact group again.

On the other hand, the 3D-NOA model does not take into account the inertia, the rest mass, of the accelerated frame S'. Actually, it is implicitly taken to be equal to zero relatively to S' and S. There is no rotation about the rest mass arrow or equivalently about the rest energy arrow E because such arrow, in S' as in S, does not exist for this simple 3D model. In relativistic dynamics, inertia depends on v/c (for an observer in S). In non-relativistic regime (i.e. $v/c \ll 1$) that dependence is so small that the motion of a uniformly accelerated body is well described by $gt^2/2$ (i.e. independent of v/c). But for $v/c \sim 1$, the effect of v/c over the inertia is to increase it such as strong corrections occur and limit the body velocity to c. The 3D-NOA model gives good results in non-relativistic regime but it is certainly unable to introduce the previous corrections simply because the inertia arrow or energy arrow doesn't exist at all.

The new arrows should be E and P_x (i.e. energy and momentum along x) because as arrows t and x they form a 4-vector (actually a 2-vector). In S', E would be the rest energy arrow or equivalently the rest mass arrow. Therefore, the most simple rotation groups that we can thing about are SO(4) and SU(3) with, as before, real angles or parameters. But SU(3) has two great advantages. First, as for SU(2), the entanglement[2] is still present and this introduces more degrees of freedom. Secondly, SU(3) can support $3^2-1 = 8$ arrows (dimensions) and then 8 angles while SO(4) gives $(4 - 1)4/2 = 6$. However, although both groups give us enough new arrows than what we need (i.e. 2), SU(3) allows to complete (partly) the 2-vectors pairs, i.e. (t, x)$\rightarrow$(t, x, y) and (E, P_x)$\rightarrow$(E, P_x, P_y), which is not the case of SO(4). Indeed, with SU(3) we have, as before, Γ_x, x, t and if we add E and P_x it remains 3 arrows which can be y, P_y and of course Γ_y so, 8 arrows. For SO(4), we have Γ_x, x, t and then E et P_x. Now it remains 1 arrow which can be what? y or P_y or Γ_y? Moreover, it has to be noted that, a priori, SU(3) keeps the same needed relations among Γ_x and (E, P_x) as the ones we had among Γ_x and (t, x) in SU(2).

A NOA model with more arrows than 3, is it the answer to our questions and the solution to our problem? Actually, we don't think so. Preliminary analysis of a SU(3) 8D-NOA model seems indicate good agreements with non-relativistic regime but fails as before in relativistic one. So, the response to our first question is: the cause of the failure is the use of a compact group with real angles in the NOA model context. Unfortunately, the NOA model cannot support imaginary angles or non-compact groups because it will lose its meaning. But, the second question is still unanswered: why this model gives us good results in non-relativistic regime? The answer is mathematical. In our first paper we introduced the assumption (85) that can be rewritten as:

$$\sin(\phi) = \beta = \mathrm{v}/\mathrm{c} \tag{86}$$

by using the first equation in (83) (i.e. first paper). ϕ is the angle about the arrow Γ_{x}. In addition, because of (86) above, the second equation in (83) (i.e. first paper) takes the form:

$$\cos(\phi) = \sqrt{1-\beta^2}\ . \tag{87}$$

As long as ϕ is small and $\beta \ll 1$ we have the approximations:

$$\sin(\phi) \ \simeq\ \phi\ , \tag{88}$$

$$\cos(\phi) \ \simeq\ 1 - \frac{\phi^2}{2} \tag{89}$$

and also:

$$\sqrt{1-\beta^2} \ \simeq\ 1 - \frac{\beta^2}{2}\ . \tag{90}$$

On the other hand, in the present context, $\phi = \mathrm{g}\tau/\mathrm{c}$ and $\mathrm{g}\tau$ is simply the velocity v. So, $\phi = \beta$. Because of that and (88)-(90) above, the equations (86) and (87) in this paper are verified in very good approximation and then, the 3D-NOA model works good (i.e. non-relativistic regime: $\beta \ll 1$). However, when ϕ becomes too large or $\beta \sim 1$ (i.e. because g is large or because τ has reached large values) approximations (88)-(90) fail and, consequently, (86) and (87) above cannot be satisfied. In that case the model fails.

6 Correction

In order to be complete, we must mention a fact which is the hypothesis (85) of the first paper, or (86) here, brings some incoherences. However, as we will show below, it is easy to correct it and this doesn't change neither the previous analysis nor the conclusion of this work. However, because of this correction, features of the 3D-NOA model found in our first paper like length contraction and time dilation are in good agreement with those of special relativity only for $\beta \ll 1$. As for the present work and for the same reason, they fail for relativistic regime $\beta \sim 1$.

The incoherence appears clearly in (85) of the present paper. Indeed, why the velocity (not the 4-velocity vector) of S' relatively to S, $\mathrm{d}x/\mathrm{d}t$, is equal to c for $\phi = \mathrm{g}\tau/\mathrm{c} = \pi/4$ and not for $\pi/2$ as it should be according to the first paper? The correction is this. We must replace (86) above by:

$$\tan(\phi) = \beta \tag{91}$$

which is consistent with (85) above. It has to be noted that, according to (83) of [2], equation (91) above is equivalent to:

$$\frac{\phi_{\mathrm{x}}^{(2')}}{\phi_{\mathrm{t}}^{(2')}} = \frac{\mathrm{v}}{\mathrm{c}} \ . \tag{92}$$

Compare with the assumption (85) of [2] which can be written as:

$$\frac{\phi_{\mathrm{x}}^{(2')}}{\phi_{\mathrm{t'}}^{(2')}} = \frac{\mathrm{v}}{\mathrm{c}} \tag{93}$$

where the equality (82) of [2], $\phi_{\mathrm{t}}^{(2)} = \phi_{\mathrm{t'}}^{(2')}$, has been used. Furthermore, because of the trigonometric relation:

$$\cos^{-2}(\phi) = 1 + \tan^2(\phi) \ , \tag{94}$$

the equation (87) of this paper is then replaced by:

$$\cos(\phi) = \frac{1}{\sqrt{1+\beta^2}} \ , \tag{95}$$

and from

$$\cos^2(\phi) + \sin^2(\phi) = 1 \tag{96}$$

we get:

$$\sin(\phi) = \frac{\beta}{\sqrt{1+\beta^2}} \ . \tag{97}$$

Now for small ϕ and $\beta \ll 1$, equations (95) and (97) become:

$$\cos(\phi) \ \simeq \ 1 - \frac{\phi^2}{2} \ = \ \frac{1}{\sqrt{1+\beta^2}} \ \simeq \ 1 - \frac{\beta^2}{2} \tag{98}$$

and

$$\sin(\phi) \ \simeq \ \phi \ = \ \frac{\beta}{\sqrt{1+\beta^2}} \ \simeq \ \beta \ . \tag{99}$$

As we can see, although we introduced a correction in order to get a consistent model, we retrieve exactly the same results as before (i.e. (88)-(89)) for the non-relativistic regime, $\beta \ll 1$.

It has to be noted that the previous correction implies that equations (76) and (77) of the first paper are wrong (i.e. they are not the good definitions of v or -v). However, equation (78) is still right because it is actually based on symmetry properties. This is why the result (82) of the first paper is still good. In addition, according to the above correction, the sin(...) in eq. (38) of the present paper must be replaced by a tan(...). But this doesn't change the result of the next equation, i.e. (39), and then neither the rest of the work.

7 Conclusion

In conclusion, for flat spacetime, we have learn in this paper about the way to produce acceleration with nodes of oriented arrows or NOA which is consistent with principles of special relativity. Doing that, we have obtained result, world line, which is in good agreement with physics only in non-relativistic regime. We also retrieved the local "lorentzian" behavior of the first paper[2] and found invariants quite similar to those of special relativity except for the sign which is also the signature of the group used. However, the results about the motion fail in relativistic regime and the likely cause of that is the use of compact group. A NOA model with more dimensions or arrows, including inertial rest mass, could not be able to get better results in the relativistic regime.

We can say that the NOA model reproduces the newtonian space-time and quite probably the classical mechanics before Einstein if more arrows would be introduced. In a sense this is surprising if we thing about the way we defined it which has nothing to do with classical mechanics. However, because it fails in relativistic regime we cannot expect getting new physical interpretation of electromagnetism and, of course, a new one for the photon. We cannot also expect getting new insights about the Einstein's gravitation and about quantum behaviors of matter.

Nevertheless, this model could be useful for didactic purposes such as, in particular, to avoid the construction of similar models by other researchers. Indeed, we must remember that this model has been logically and literally built according to Ummite texts about IBOZOO UU (see first paper). The three main concepts introduced by those texts, OAWOO, IOAWOO and IBOZOO UU, have been represented respectively in this model by our usual concepts of "arrows" or "axis", "angles" and by a junction of these axis; NOA. The use of our angles needs the use of our rotations and our rotation groups. But it is clear from the results of this paper that our simple concepts of axis and angles and then rotations have nothing to do with theirs. With regard to Ummite file, this is the major contribution of these two papers; they don't show what the Ibozoo uu are but they clearly show what they aren't.

Nota Bene. The three previous paragraphs of this conclusion have been written before we came aware about the transformation and results of the appendix. So, may be a little hope is still allowed for the NOA model...

8 Appendix

8.1 Getting Exactly Special Relativity From the NOA Model by a Simple Mathematical Transformation

It is quite simple to get exactly all results of special relativity (i.e. those mentioned in these two papers) from the results of NOA model by using a simple transformation. This transformation is to replace ϕ and c in the expressions of the NOA model by iϕ and $-$ic respectively:

$$\mathrm{c} \rightarrow -\mathrm{ic} \qquad \phi \rightarrow \mathrm{i}\phi \tag{100}$$

where i is the imaginary symbol (i.e. $\sqrt{-1}$) and ϕ is the (real) angle about Γ_{x}. In addition to this transformation, we must take into account the mathematical relations:

$$\cos(\mathrm{i}\phi) = \cosh(\phi) \tag{101}$$

$$\sin(\mathrm{i}\phi) = \mathrm{i}\sinh(\phi) \ . \tag{102}$$

Before going further, let's introduce some well known facts from special relativity. For instance, the Lorentz transformations (28) given in our first paper can be rewritten as:

$$\begin{aligned} x &= \gamma(x' + \mathrm{v}t') \\ \mathrm{c}t &= \gamma(\mathrm{c}t' + \beta x') \end{aligned} \tag{103}$$

where $\beta = \mathrm{v}/\mathrm{c}$ and:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \ . \tag{104}$$

The expressions in (103) can also be written as:

$$\begin{aligned} x &= x' \cosh(\phi) + \mathrm{c}t' \sinh(\phi) \\ \mathrm{c}t &= \mathrm{c}t' \cosh(\phi) + x' \sinh(\phi) \end{aligned} \tag{105}$$

where ϕ is a real number which is so-called the "boost parameter" and:

$$\cosh(\phi) = \gamma = \frac{1}{\sqrt{1-\beta^2}} \ , \tag{106}$$

$$\sinh(\phi) = \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}} \ . \tag{107}$$

From the two previous ones we get:

$$\tanh(\phi) = \beta \ . \tag{108}$$

Now, using the transformation (100) and the equalities (101)-(102) it is easy to get the special relativity results from those of the NOA model:

NOA Model	$\phi \to i\phi$, $c \to -ic$	Special Relativity
(95) $\cos(\phi) = \frac{1}{\sqrt{1+\beta^2}}$	$\to$	(106) $\cosh(\phi) = \frac{1}{\sqrt{1-\beta^2}}$
(97) $\sin(\phi) = \frac{\beta}{\sqrt{1+\beta^2}}$	$\to$	(107) $\sinh(\phi) = \frac{\beta}{\sqrt{1-\beta^2}}$
(91) $\tan(\phi) = \beta$	$\to$	(108) $\tanh(\phi) = \beta$
(85) $\mathrm{d}x/\mathrm{d}t = \mathrm{c}\tan(\mathrm{g}\tau/\mathrm{c})$	$\to$	(84) $\mathrm{d}x/\mathrm{d}t = \mathrm{c}\tanh(\mathrm{g}\tau/\mathrm{c})$
(47) $(x - \mathrm{c}^2/\mathrm{g})^2 + (\mathrm{c}t)^2 = \mathrm{c}^4/\mathrm{g}$	$\to$	(80) $(x + \mathrm{c}^2/\mathrm{g})^2 - (\mathrm{c}t)^2 = \mathrm{c}^4/\mathrm{g}$
(53) $\tilde{\mathbf{u}}^2 = (\tilde{\mathrm{u}}^1)^2 + (\mathrm{c}\tilde{\mathrm{u}}^0)^2 = \mathrm{c}^2$	$\to$	(67) $\mathbf{u}^2 = (\mathrm{u}^1)^2 - (\mathrm{cu}^0)^2 = -\mathrm{c}^2$
(54) $\tilde{\mathbf{a}}^2 = (\tilde{\mathrm{a}}^1)^2 + (\mathrm{c}\tilde{\mathrm{a}}^0)^2 = \mathrm{g}^2$	$\to$	(68) $\mathbf{a}^2 = (\mathrm{a}^1)^2 - (\mathrm{ca}^0)^2 = \mathrm{g}^2$
(55) $\tilde{\mathbf{a}} \cdot \tilde{\mathbf{u}} = \tilde{\mathrm{a}}^1\tilde{\mathrm{u}}^1 + \mathrm{c}^2\tilde{\mathrm{a}}^0\tilde{\mathrm{u}}^0 = 0$	$\to$	(69) $\mathbf{a} \cdot \mathbf{u} = \mathrm{a}^1\mathrm{u}^1 - \mathrm{c}^2\mathrm{a}^0\mathrm{u}^0 = 0$
(64) $\mathrm{ds}^2 = (\mathrm{d}x)^2 + (\mathrm{cd}t)^2$	$\to$	(70) $\mathrm{ds}^2 = (\mathrm{d}x)^2 - (\mathrm{cd}t)^2$

Furthermore let's take, for instance, the result (124) in our first paper:

$$\begin{aligned} \phi_{\mathrm{x}}^{(1')} &= \phi_{\mathrm{t}'}^{(1')} \sin(\phi) + \phi_{\mathrm{x}'}^{(1')} \cos(\phi) \\ \phi_{\mathrm{t}}^{(1')} &= \phi_{\mathrm{t}'}^{(1')} \cos(\phi) - \phi_{\mathrm{x}'}^{(1')} \sin(\phi) \end{aligned} \tag{109}$$

where the equality $\phi_{\mathrm{t}}^{(2)} = \phi_{\mathrm{t}'}^{(2')} = \phi_{\mathrm{t}'}^{(1')}$[2] has been taken. Using the universal function $\mathcal{F}_{\mathrm{x}}$ on both sides of these equations and taking into account the properties of this function (see [2] including the fact that $\mathcal{F}_{\mathrm{x}} = \mathrm{c}\mathcal{F}_{\mathrm{t}}$) and the correction involving (95) and (97), we get:

$$\begin{aligned} \mathcal{F}_{\mathrm{x}}(\phi_{\mathrm{x}}^{(1')}) &= \frac{1}{\sqrt{1+\beta^2}}\Big(\mathrm{v}\mathcal{F}_{\mathrm{t}}(\phi_{\mathrm{t}'}^{(1')}) \;+\; \mathcal{F}_{\mathrm{x}}(\phi_{\mathrm{x}'}^{(1')})\Big) \\ \mathrm{c}\mathcal{F}_{\mathrm{t}}(\phi_{\mathrm{t}}^{(1')}) &= \frac{1}{\sqrt{1+\beta^2}}\Big(\mathrm{c}\mathcal{F}_{\mathrm{t}}(\phi_{\mathrm{t}'}^{(1')}) \;-\; \beta\mathcal{F}_{\mathrm{x}}(\phi_{\mathrm{x}'}^{(1')})\Big) \end{aligned} \tag{110}$$

instead of (125) of the first paper. According to the natural definitions:

$$\begin{aligned} \mathcal{F}_{\mathrm{x}}(\phi_{\mathrm{x}}^{(1')}) &= x \\ \mathcal{F}_{\mathrm{x}}(\phi_{\mathrm{x}'}^{(1')}) &= x' \\ \mathcal{F}_{\mathrm{t}}(\phi_{\mathrm{t}}^{(1')}) &= t \\ \mathcal{F}_{\mathrm{t}}(\phi_{\mathrm{t}'}^{(1')}) &= t' \end{aligned} \tag{111}$$

and the transformation (100), equations in (110) reduce to:

$$x = \gamma(x' + \mathrm{v}t')$$

$$\mathrm{i}\Big[ct \;=\; \gamma(\mathrm{c}t' \;+\; \beta x')\Big] \tag{112}$$

which are identical to those of (103).

It is amazing to retreive special relativity from the NOA model by using the mathematical transformation (100). Unfortunately, up to now, this transformation has no physical meaning.

9 Figures

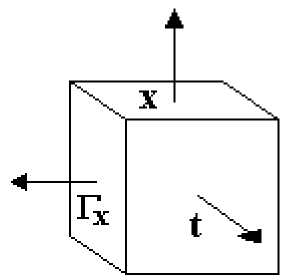

Figure 1: A 3D-NOA (i.e. 3 arrows) geometric representation in which three perpendicular arrows, t, x and Γ_x, are shown. The little cube reminds that there is no identification to do in between these arrows and the usual axis of any cartesian coordinates system. There is no linear scale along each arrows conversely to cartesian axis. A NOA is simply a junction of oriented arrows. The length of arrows is irrelevant. So, it is arbitrary fixed to one. Arrows do not have to cross at a unique point (i.e. the cube center).

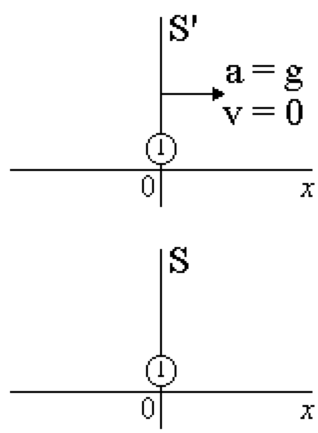

Figure 2: Two reference frames. S is inertial and S' is accelerated along the space coordinate x of S. The initial velocity v of S' is zero and its acceleration "a" is a constant g. Clocks located at their respective origin mark zero when those origins coincide.

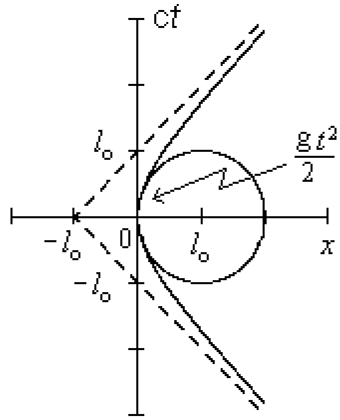

Figure 3: Circular and hyperbolic curves defined by eqs. (47) and (80) respectively. Dashed lines represent the asymptotes of the hyperbola. They represent also some photons world lines. $l_o = c^2/g$. The region where circular and hyperbolic coincide is the one where the non-relativistic result $gt^2/2$ works.

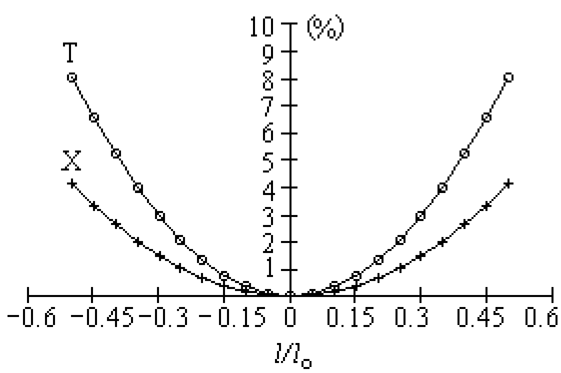

Figure 4: Curves giving the relative discrepancies, $X = (x_{\rm hyp} - x_{\rm circ})/x_{\rm hyp}$ and $T = (t_{\rm hyp} - t_{\rm circ})/t_{\rm hyp}$, as function of l/l_o. $x_{\rm hyp}$ and $t_{\rm hyp}$ are the values of x and t given by eqs. (78) and (79) respectively. On the other hand, $x_{\rm circ}$ and $t_{\rm circ}$ are those given respectively by (45) and (46). $l_o = c^2/g$ and $l = c\tau$.

10 Bibliography

References

[1] e-mail: enedictus@hotmail.com

[2] Servant, B., *I. Les Ibozoo uu et L'Espace-Temps Relativiste*, 4 juin 2003. http://www.ummo-sciences.org/activ/science/ibozoo/ibozoo.htm

[3] Because of the needed correction introduced in §6 of this paper, length contraction and time dilation are retrieved only for non-relativistic regime: $v/c \ll 1$.

[4] Because of the correction mentioned in [3], velocity v reaches c not when the angle about the arrow Γ_x is $\pi/2$ but when it is $\pi/4$. Therefore, conversely to the claim made in the first paper, the permutation or inversion in between x and t arrows is an idea that we cannot use to characterize the limit c.

[5] At first order of the small angles.

[6] Misner, C.W., Thorne, K.S. and Wheeler, J.A., *Gravitation*, ed. W.H. Freeman and Company, (1973).

[7] See: Stanford Linear Accelerator Center. http://www.slac.stanford.edu/